1.-

(a) For which value of $k$ is the following a probability distribution for the random variable $X$

$$P(X = x) = \frac{x + k}{10}, \quad x = 0, 1, 2, 3$$

$$P(0) = \frac{0 + k}{10}, \quad P(1) = \frac{1 + k}{10}, \quad P(2) + P(3) = 1$$

\[\begin{align*}
0 + k &= 10 \\
1 + k &= 10 \\
2k &= 10 \\
4k &= 10
\end{align*}\]

\(\Rightarrow\) \(4k = 10\) \(\Rightarrow\) \(k = 1\)

(b) Find $E(X) = \mu$ and $\sigma$

<table>
<thead>
<tr>
<th>$X$</th>
<th>$P(X)$</th>
<th>$xP(X)$</th>
<th>$x^2P(X)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>0.2</td>
<td>0.2</td>
<td>0.2</td>
</tr>
<tr>
<td>2</td>
<td>0.3</td>
<td>0.6</td>
<td>1.2</td>
</tr>
<tr>
<td>3</td>
<td>0.4</td>
<td>1.2</td>
<td>3.6</td>
</tr>
</tbody>
</table>

\[\begin{align*}
\frac{0 + 1 + 2 + 3}{4} &= \mu \\
\frac{0^2 + 1^2 + 2^2 + 3^2}{4} &= \alpha
\end{align*}\]

\[\begin{align*}
\sigma^2 &= E[X^2] - \mu^2 \\
&= 5 - 2^2 \\
&= 5 - 4 \\
&= 1 \\
\sigma &= \sqrt{1} \\
&= 1
\end{align*}\]

2.- Three out of Four Cats prefer mice to cat food. If 10 cats are selected randomly, what is the probability that:

(a) Exactly 6 of them prefer mice to cat food?

$$P(X = 6) = \binom{10}{6} (0.75)^6 (0.25)^4$$

(b) At least one will prefer mice to cat food.

$$P(X \geq 1) = 1 - P(X = 0)$$

$$= 1 - \binom{10}{0} (0.75)^0 (0.25)^{10}$$
(c) At most two prefer mice to cat food?

\[
P(X \leq 2) = P(X=0) + P(X=1) + P(X=2)
= 10C_0 (0.75)^0 (0.25)^10 + 10C_1 (0.75)^1 (0.25)^9 + 10C_2 (0.75)^2 (0.25)^8
\]

(d) How many cats are expected to prefer mice over cat food?

\[
\mu = np = 10 \times 0.75 = 7.5 \approx 8 \text{ cats}
\]

3.- Suppose that the amount of caffeine in energy drinks is normally distributed, with a mean of 81 mg. If 3% of the drinks have more than 100 mg, what is the standard deviation of this distribution?

\[X = \text{amount of caffeine} \]

\[
P(X > 100) = 0.03
\]

\[
Z = \frac{X - \mu}{\sigma}
\]

\[
\Rightarrow \sigma = \frac{X - \mu}{Z} = \frac{100 - 81}{1.98} \approx 10.1
\]

4.- Suppose that the speed of cars in the freeway 405 is normally distributed with a mean speed of 75 mph and a standard deviation of 10 mph. What is the probability that

(a) A randomly chosen car will travel faster than 90 mph?

\[
\mu = 75, \sigma = 10,
\]

\[
Z = \frac{90 - 75}{10} = 1.5
\]

\[
P(X > 90) = P(Z > 1.5) = 0.0668
\]
(b) A car will travel between 65 mph and 90 mph?

\[ P(65 < X < 90) = P(-1 < z < 1.5) \]
\[ = 0.3413 + 0.4332 \]
\[ = 0.7745 \]

OR

\[ \text{normal cdf}(-1, 1.5) \approx 0.7745 \]

(c) Now suppose that you pick a sample of 25 cars, what is the probability that the sample mean will exceed 77 mph? Does CLT applies? explain?

\[ z = \frac{77 - 75}{10/\sqrt{25}} = 1 \]

The CLT applies because the distribution is normal.

5.- A poll is conducted to see what proportion of people believe in Santa Claus. Out of 200 people 120 said that they believe in Santa Claus.

(a) Find a 99% C.I for \( p \).

\[ \alpha = 0.01, \ n = 200, \ \hat{p} = 0.6 \]

\[ \hat{p} \pm E = 0.6 \pm 0.09 \]

\[ 0.51 < p < 0.69 \]

Conclusion: you are 99% confident that between 51% and 69% of people believe in Santa.

(b) How large of a sample is needed to have an error of 2%?

\[ n = \left( \frac{Z_{\alpha/2}}{E} \right)^2 \hat{p} (1-\hat{p}) = \left( \frac{2.58}{0.02} \right)^2 (0.6)(0.4) \]
\[ = 3993.8 \]
\[ \approx 3994 \]
6.- The following data represents the monthly payment (in thousands) that employees get at company TUPS: 5,2,4,3,10, and 4

\[ n = 6 \]

(a) Find \( \bar{X} \) and \( S \):

\[ \bar{X} = 4.7 \]
\[ S = 2.8 \]

\[ S^2 = \frac{\sum x^2 - (\sum x)^2/n}{n-1} \]

(b) Construct a 95 percent C.I. for the above data.

\[ \bar{X} = 0.025, \text{ to } 0.025, S = 2.571 \]

\[ E = 2.571 \left( \frac{2.8}{\sqrt{6}} \right) \approx 3 \]

\[ \bar{X} \pm E \Rightarrow 4.7 \pm 3 \]

7.- A sample of 100 people was conducted to see how many cups of coffee (per day) people buy in starbucks. The sample has a mean of 4 cups and a standard deviation of 2.5 cups.

(a) Find a 95 percent C.I. for the true mean \( \mu \).

\[ \bar{X} = 4, \sigma = 2.5, n=100 \]

\[ \frac{\sigma}{\sqrt{n}} = 0.025 = 1.96 \]

\[ E = 1.96 \left( \frac{2.5}{\sqrt{100}} \right) \approx 0.5 \]

\[ \bar{X} \pm E \Rightarrow 4 \pm 0.5 \]

(b) What is the margin error?

\[ E = 0.49 \approx 0.5 \]
8.- Find a 99% confidence interval for $\sigma$, when $n = 17$, $S^2 = 37$

\[
\frac{(n-1)S^2}{X^2_R} < \sigma^2 < \frac{(n-1)S^2}{X^2_L}
\]

\[
\frac{(16)(37)}{34.267} < \sigma^2 < \frac{(16)(37)}{5.142}
\]

\[
17.3 < \sigma^2 < 115.1 \implies 4.2 < \sigma < 10.7
\]

10.- T/F section.

(a) For any random sample $P(\chi^2 < \chi^2_{L}) = P(\chi^2 > \chi^2_{R})$. 

(b) For any random variable $X$ the variance is np(1-p).

(c) $P(Z > a) = 1 - P(Z < a)$ \text{ True}

(d) Every random variable $X$ has a probability distribution $P(X = k)$

(e) A binomial random variable with $p = 0.4$ and $n = 20$ has a mean of 8. \text{ True}

(f) $P(Z > -1) = P(Z < 1)$ \text{ True} \quad \mu = np = 20(0.4) = 8 \quad z = \frac{20 - \mu}{\sigma}

(g) $Z_{\alpha/2} > Z_{\alpha}$. \text{ True} \quad \alpha = 0.05

(h) $\bar{X}$ is normally distributed even if the sample size is less than 30? \text{ False}

(i) For any normal random variable $X$, $P(X > k) = P(Z > \frac{k - \mu}{\sigma})$ \text{ True}